

# $\lambda$ -symmetries and Jacobi Last Multiplier

M.C. Nucci\*

Dipartimento di Matematica e Informatica,  
Università degli Studi di Perugia & INFN Sezione Perugia,  
06123 Perugia, Italy

D. Levi†

Dipartimento di Ingegneria Elettronica,  
Università degli Studi Roma Tre & INFN Sezione Roma TRE,  
Via della Vasca Navale 84, 00146 Roma, Italy

November 8, 2011

## Abstract

We show that  $\lambda$ -symmetries can be algorithmically obtained by using the Jacobi last multiplier. Several examples are provided.

Keywords: Ordinary differential equations, Lie group analysis

PACS 02.30.Hq, 02.20.Sv

## 1 Introduction

Lie group analysis is the most powerful general tool to find the solution of Ordinary Differential Equations (ODEs). However an ODE of  $n$ th order does not always admit Lie point symmetries. Moreover this Lie group analysis is useless when applied to  $n$  equations of first order because they admit an infinite number of symmetries, and there is no systematic way to find even a one-dimensional Lie symmetry algebra. One may try to derive an admitted  $n$ -dimensional solvable Lie symmetry algebra by making an ansatz on the form of its generators but when successful (rarely) it is just a lucky guess. However, in [26] it has been shown that any system of  $n$  equations of first order can be transformed into an equivalent system where at least one of the equations is of second

---

\*e-mail: [nucci@unipg.it](mailto:nucci@unipg.it)

†email: [levi@roma3.infn.it](mailto:levi@roma3.infn.it)

order. Then the admitted Lie symmetry algebra is no longer infinite-dimensional, and nontrivial symmetries of the original system may be retrieved. This idea has been successfully applied in several instances ([26], [36], [31], [27], [16], [28], [5], [8]).

In [29] another method was devised. It uses the Jacobi last multiplier ([11], [12], [13], [14], [17], [18], [1], [37]) to transform a system of  $n$  first-order equations into an equivalent system of  $n$  equations where one of the equations is of second order, namely the order of the system is raised by one. In [29], among other examples, the method was successfully applied to the second-order equation [15][Ch. 6, 542ff]:

$$y'' = \frac{y'^2}{y} + f'(t)y^{p+1} + pf(t)y'y^p, \quad (1)$$

where  $p \neq 0$  is a real constant and  $f \neq 0$  is an arbitrary function of the independent variable  $t$ . This equation does not possess Lie point symmetries for general  $f(t)$  and yet is trivially integrable [7]. In [29] the introduction of the Jacobi last multiplier led to an equivalent system of two equations, one of first order and the other of second order, which admits enough Lie symmetries in order to integrate it by quadrature.

In [20] and [21] Muriel and Romero introduced the so-called  $\lambda$ -symmetries which were later included into the telescopic symmetries by Pucci and Saccomandi [35]. Again as in the case of Lie point symmetries of first-order equations the real problem is to find solutions of the determining equations. This problem has been clearly stated by Pucci and Saccomandi on page 6154 of their article.

In [4] and [6] Cicogna, Gaeta and Morando provided a geometrical characterization of  $\lambda$ -symmetries for both ordinary and partial differential equations. However, in either paper they did not address the problem of solving the determining equation, they just obtain a  $\lambda$ -symmetry by guesswork.

In [2] Catalano Ferraioli showed that  $\lambda$ -symmetries correspond to a special type of nonlocal symmetries and in a final remark observed that also telescopic symmetries can be recovered by nonlocal symmetries. Yet there is much guesswork involved in order to find nonlocal symmetries, and on page 5485, Remark 2, Catalano Ferraioli wrote: “However, we note that the problem of determining the general solution of (11)<sup>1</sup>, should be at least as difficult as solving the given ODE. Therefore in practice it could not be so easy to determine such correspondence (see example 3)”.

In [30] the method described in [29] was applied to example 3 in [2], i.e

$$y'' = \frac{y'^2}{y} + \left(y + \frac{t}{y}\right)y' - 1, \quad (2)$$

that is a second-order Painlevé-type equation [34], which does not admit any Lie point symmetries and is a particular case<sup>2</sup> of Painlevé XIV equation [9], i.e.:

$$y'' = \frac{y'^2}{y} + \left(Q(t)y + \frac{S(t)}{y}\right)y' + Q'(t)y^2 - S'(t), \quad (3)$$

---

<sup>1</sup>It is a linear first-order partial differential equation.

<sup>2</sup>It corresponds to assume  $Q(t) = 1$  and  $S(t) = t$ .

which does not possess any Lie point symmetry for arbitrary  $Q(t), S(t)$  although it has a Riccati-type first integral, i.e.:

$$\frac{y' - Q(t)y^2 + S(t)}{y} = a_1, \quad (4)$$

with  $a_1$  an arbitrary constant. In [30] the introduction of the Jacobi last multiplier into equation (2) led to a system of three equations, two of first order and one of second order. This system admits a three-dimensional solvable Lie symmetry algebra and therefore can be reduced to a Riccati equation that can be integrated in terms of Airy functions. Thus a new first integral different from (4) was found, i.e.

$$a_2 = \frac{(y^2 + y' + t)\text{AiryAi}(\xi) + 2y\text{AiryAi}(1, \xi)}{(y^2 + y' + t)\text{AiryBi}(\xi) + 2y\text{AiryBi}(1, \xi)}, \quad \xi = \frac{y^4 + 2(t - y')y^2 + (t + y')^2}{4y^2}. \quad (5)$$

with  $a_2$  an arbitrary constant. Then, combining the two first integrals (4)– with  $Q(t) = 1$  and  $S(t) = t$  – and (5), one gets the general solution of equation (2) in implicit form, i.e.

$$\frac{(2y + a_1)\text{AiryAi}(\xi) + 2\text{AiryAi}(1, \xi)}{(2y + a_1)\text{AiryBi}(\xi) + 2\text{AiryBi}(1, \xi)} = a_2, \quad \xi = t + \frac{a_1^2}{4}. \quad (6)$$

This suggests that there might be a direct link between the Jacobi last multiplier and  $\lambda$ -symmetries.

In the present paper we show that the introduction of the Jacobi last multiplier allows to find  $\lambda$ -symmetries algorithmically.

In the next Section, we briefly recall the connection between nonlocal symmetries and  $\lambda$ -symmetries as shown by Catalano Ferraioli in [2], and some essential properties of the Jacobi last multiplier. Then we show how to use the Jacobi last multiplier to find  $\lambda$ -symmetries. Recently in [23] Muriel and Romero have classified all the  $\lambda$ -symmetries of any second-order ODE through an equivalence relationship and proved that two  $\lambda$ -symmetries lead to functionally independent first integral if and only if they are in different equivalence classes. This important result allows us to discriminate the more than one  $\lambda$ -symmetry that we found in some of the examples. In Section 3 we apply our method based on the Jacobi last multiplier and either recover known or find new  $\lambda$ -symmetries of several differential equations of second order: equation (1) as given in [7], equations Painlevé V, XIV (3), XV, and XVI as given in [9]<sup>3</sup>, and examples 4 and 5 as given in [3]. Section 4 is devoted to some conclusions.

## 2 $\lambda$ -symmetries and Jacobi Last Multiplier

Let us consider an  $n^{\text{th}}$ -order ODE:

$$y^{(n)} = f(t, y, y', y'', \dots, y^{(n-1)}), \quad (7)$$

---

<sup>3</sup>In particular we recover the known first integrals [9].

where by an apex we mean the order of differentiation. In [2] a nonlocal interpretation of  $\lambda$ -symmetries was given. There Catalano Ferraioli has shown that seeking  $\lambda$ -symmetries of eq. (7) is equivalent to add to equation (7) the equation

$$\omega' = \lambda \quad (8)$$

for the new field  $\omega(t)$  with  $\lambda = \lambda(t, y, y', y'', \dots, y^{(n-1)})$ . The symmetries of (7, 8) are obtained by considering the infinitesimal operator:

$$Y = \tilde{\tau}(t, y, \omega)\partial_t + \tilde{\eta}(t, y, \omega)\partial_y + \tilde{\xi}(t, y, y', y'', \dots, y^{(n-1)}, \omega)\partial_\omega \quad (9)$$

with the constraint

$$[Y, \partial_\omega] = Y. \quad (10)$$

Under the constraint (10) we get  $Y = e^\omega[X + \xi\partial_\omega]$  where

$$X = \tau(t, y)\partial_t + \eta(t, y)\partial_y \quad (11)$$

and  $\xi = \xi(t, y, y', y'', \dots, y^{(n-1)})$ . The prolongation of  $X$  is

$$\begin{aligned} \text{pr}X &= \tau(t, y)\partial_t + \eta(t, y)\partial_y + \eta^{(1)}(t, y, y', y'', \dots, y^{(n-1)})\partial_{y'} \\ &\quad + \eta^{(2)}(t, y, y', y'', \dots, y^{(n-1)})\partial_{y''} + \dots \end{aligned} \quad (12)$$

with

$$\eta^{(n+1)} = \left[ (D_t + \lambda)\eta^{(n)} - y'(D_t + \lambda)\tau \right] \quad (13)$$

where  $D_t = \partial_t + \sum_{k=0}^n y^{(k+1)}\partial_{y^{(k)}}$ ,  $y^{(0)} \equiv y$ , and  $\eta^{(0)} \equiv \eta$ . A  $\lambda$ -symmetry of equation (7) is any solution of the determining equation

$$\text{pr}X \left( y^{(n)} - f(t, y, y', y'', \dots, y^{(n-1)}) \right) \big|_{y^{(n)}=f} = 0, \quad (14)$$

depending on the three unknowns  $\lambda$ ,  $\tau$  and  $\eta$  and therefore is highly undetermined. In [23] proved that any  $\lambda$ -symmetry can be put in an evolution form as a translation in  $y$ , i.e. with  $\tau = 0$  and  $\eta = 1$ . Hence equation (14) becomes determined since it has only one unknown, but it is still difficult to solve it since it is a nonlinear partial differential equation in the unknown  $\lambda$ .

If equation (7) is transformed into an equivalent system of first-order equations, i.e.

$$w'_i = W_i(t, w_1, \dots, w_n), \quad (15)$$

then its Jacobi last multiplier  $M$  is obtained by solving the following differential equation

$$\frac{d \log(M)}{dt} + \sum_{i=1}^n \frac{\partial W_i}{\partial w_i} = 0, \quad (16)$$

i.e.

$$M = \exp \left( - \int \sum_{i=1}^n \frac{\partial W_i}{\partial w_i} dt \right). \quad (17)$$

In [29] all the properties of the Jacobi last multiplier are listed. There three strategies were proposed with the purpose of finding Lie symmetries of any system (15):

1. Eliminate – if possible – each of the variables  $w_i$  in order to obtain an equivalent  $n$ -order system which contains a single equation of second order and  $n-2$  equations of first order. The admitted Lie symmetry algebra is no longer infinite-dimensional and the Lie group analysis can be usefully applied ([26], [36], [5]). From this strategy we also get first integrals ([19], [27], [16]).
2. Decrease the order of system (15) by one choosing one of the variables  $w_i$  as the new independent variable. Then apply either Strategy 1 or 3 in this list. For example, if  $w_1 \equiv y$  is the new independent variable, then system (15) becomes

$$\frac{dw_k}{dy} = \frac{W_k}{W_1} \equiv \Omega_k(y, w_2, \dots, w_n) \quad (k = 2, n) \quad (18)$$

This method has been applied in several examples (e.g. [31], [16], [28]) since its first instance, the Kepler problem [26].

3. Increase the order using the transformation suggested by the Jacobi last multiplier, i.e. introducing a new dependent variable  $R$  such that

$$\frac{dR}{dt} = \sum_{i=1}^n \frac{\partial W_i}{\partial w_i} \quad (19)$$

and eliminating – if possible – each  $w_i$  which appears in (19). Then system (15) reduces to a system given by a single second-order ODE and  $n-1$  first-order equations. If a new independent variable was chosen (Strategy 2), say  $w_1 \equiv y$ , then  $R$  will satisfy the following equation:

$$\frac{dR}{dy} = \sum_{k=2}^n \frac{\partial \Omega_k}{\partial w_k} \quad (20)$$

and then the strategy goes as above.

In [29] Strategy 3 was applied to find Lie symmetries of several systems.

The similarity between Strategy 3 and the nonlocal approach to  $\lambda$ -symmetries as given by Catalano Ferraioli suggests to search for  $\lambda$ -symmetries such that

$$\omega' = \lambda = \sum_{i=1}^n \frac{\partial W_i}{\partial w_i}. \quad (21)$$

This implies, when feasible, that  $\omega = \log(1/M)$ . In fact this connection cannot be made if the divergence of the system (15), namely  $Div \equiv \sum_{i=1}^n \frac{\partial W_i}{\partial w_i}$ , is zero, since then any Jacobi last multiplier is a first integral of (15) and therefore such is  $\omega$ .

However there are many ways in which a system of  $n$  first-order equations can be written as a single equation of  $n$ th order and viceversa. The Jacobi last multiplier is then different and if one way yields  $Div = 0$ , another way may yield  $Div \neq 0$ . In particular the following system of two first order equations

$$w'_1 = W_1(t, w_2), \quad w'_2 = W_2(t, w_1) \quad (22)$$

has  $Div = 0$  and therefore  $M_{[w_1, w_2]} = 1$  is one of its Jacobi last multipliers. If we derive  $w_1$  from the second equation, i.e.  $w_1 = \overline{W}_2(t, w'_2)$ , then an equivalent second-order ODE is obtained, i.e.

$$w''_2 = \frac{W_1(t, w_2) - \frac{\partial}{\partial t} \overline{W}_2(t, w'_2)}{\frac{\partial}{\partial w'_2} \overline{W}_2(t, w'_2)} \quad (23)$$

which has  $Div \neq 0$  since<sup>4</sup>

$$M_{[w_2]} = M_{[w_1, w_2]} \frac{\partial(w_1, w_2)}{\partial(w_2, w'_2)} = M_{[w_1, w_2]} \begin{vmatrix} 0 & \frac{\partial}{\partial w'_2} \overline{W}_2(t, w'_2) \\ 1 & 0 \end{vmatrix} = \frac{\partial}{\partial w'_2} \overline{W}_2(t, w'_2), \quad (24)$$

i.e. its Jacobi last multiplier cannot be a constant. An illustrative example of such an instance is the following system studied in [33]

$$r'_1 = b \exp(r_2) + a \quad r'_2 = B \exp(r_1) + A \quad (25)$$

which has obviously  $Div = 0$ . Following [33] we can transform system (25) into an equivalent second-order ordinary differential equation by eliminating  $r_1$ . In fact from the second equation in (25) one gets

$$r_1 = \log \left( \frac{r'_2 - A}{B} \right), \quad (26)$$

---

<sup>4</sup>This is one of the known properties of the Jacobi last multiplier [1], [29]: given a non-singular transformation of variables

$$\tau : (w_1, w_2, \dots, w_n) \longrightarrow (r_1, r_2, \dots, r_n),$$

then the last multiplier  $M_{[r]}$  of the new system  $r'_i = R_i(t, r_1, \dots, r_n)$  is given by:

$$M_{[r]} = M_{[w]} \frac{\partial(w_1, w_2, \dots, w_n)}{\partial(r_1, r_2, \dots, r_n)}.$$

and the equivalent second-order equation in  $r_2$  is the following

$$r_2'' = -\left(b \exp(r_2) + a\right)(A - r_2'). \quad (27)$$

which has  $Div = b \exp(r_2) + a \neq 0$ .

As a final remark we recall that recently [23] Muriel and Romero have proved the equivalence between two  $\lambda$ -symmetries of a second-order ordinary differential equation

$$y'' = \phi(t, y, y'). \quad (28)$$

Assuming that  $\lambda_1$  and  $\lambda_2$  yield two  $\lambda$ -symmetries  $X_1 = \tau_1(t, y)\partial_t + \eta_1(t, y)\partial_y$  and  $X_2 = \tau_2(t, y)\partial_t + \eta_2(t, y)\partial_y$  of equation (28), respectively, they are equivalent if and only if

$$Q_1(A + \lambda_2)(Q_2) - Q_2(A + \lambda_1)(Q_1) = 0, \quad (29)$$

where  $Q_i = \eta_i - y'\tau_i$ , ( $i = 1, 2$ ) and  $A = \partial_t + y'\partial_y + \phi(t, y, y')\partial_{y'}$  is the vector field associated with (28). This equivalence will be used in the next Section 3 when we obtain more than one  $\lambda$ -symmetry.

### 3 Examples

In the following examples we denote by  $\lambda_k$  the  $\lambda$ -symmetries presented by Muriel, Romero, Catalano Ferraioli and Morando in the references [20], [21], [3] and by  $\lambda_J$  the  $\lambda$ -symmetry that we find by using formula (21) once we rewrite (7) as (15).

#### 3.1 Equation (1)

In [20] a  $\lambda$ -symmetry of equation (1) was determined, i.e.

$$X^{(\lambda_k)} = \partial_y \quad \text{with} \quad \lambda_k = py^p f(t) + y'/y. \quad (30)$$

The divergence of equation (1) yields

$$\lambda_J = py^p f(t) + 2\frac{y'}{y}. \quad (31)$$

If we put  $\lambda_J$  into (12) then the solution of the determining equations (14) yields two  $\lambda$ -symmetries, i.e.

$$X_1^{(\lambda)} = \frac{1}{y}\partial_y, \quad X_2^{(\lambda)} = \frac{1}{y^2}\partial_t + y^{p-1}f(t)\partial_y. \quad (32)$$

The first prolongation of  $X_1^{(\lambda)}$ , i.e.

$$\text{pr}X_1^{(\lambda)} = X_1^{(\lambda)} + \left(py^{p-1}f(t) + \frac{y'}{y^2}\right)\partial_{y'} \quad (33)$$

yields the first-order invariants

$$y_1 = -y^p f(t) + \frac{y'}{y}, \quad t_1 = t \quad (34)$$

that replaced into equation (1) generate the first-order equation

$$\frac{dy_1}{dt_1} = y' = 0, \quad (35)$$

as obtained in [20]. The first prolongation of  $X_2^{(\lambda)}$ , i.e.

$$\text{pr}X_2^{(\lambda)} = X_2^{(\lambda)} + (y^{2p-1}f^2(t)p + y^{p-1}f'(t) + y^{p-2}y'f(t))\partial_{y'} \quad (36)$$

yields the first-order invariants

$$y_2 = F^{\frac{1}{p}}(t)y' + \frac{F'(t)}{pF(t)}, \quad t_2 = \frac{1}{y^p} + p \int f(t) dt \quad (37)$$

where  $F(t) = t_2 - p \int f(t) dt$ , that replaced into equation (1) generate the same invariant as  $X_1^{(\lambda)}$ .

This result should not be a surprise since the two  $\lambda$ -symmetries (32) and the  $\lambda$ -symmetry  $\lambda_k$  found in [20] are all equivalent as defined by Muriel and Romero in [23]. In fact substituting  $X_1^{(\lambda)}$  and  $X_2^{(\lambda)}$  – that have the same  $\lambda_J$  (31) – into (29) yields:

$$\begin{aligned} & Q_1(A + \lambda_J)(Q_2) - Q_2(A + \lambda_J)(Q_1) = Q_1(A)(Q_2) - Q_2(A)(Q_1) \\ &= \frac{1}{y} \left[ y^{p-1}f'(t) + y' \left( (p-1)y^{p-2}f(t) + 2\frac{y'}{y^3} \right) - \frac{1}{y^2} \left( \frac{y'^2}{y} + f'(t)y^{p+1} + pf(t)y'y^p \right) \right] \\ & \quad + \left( y^{p-1}f(t) - \frac{y'}{y^2} \right) \frac{y'}{y^2} = 0 \end{aligned} \quad (38)$$

since  $Q_1 = 1/y$  and  $Q_2 = y^{p-1}f(t) - y'/y^2$ . Also substituting  $X_1^{(\lambda)}$  and  $X^{(\lambda_k)}$  and their corresponding  $\lambda_J$  and  $\lambda_k$  into (29) yields:

$$Q_1(A + \lambda_k)(Q_k) - Q_k(A + \lambda_J)(Q_1) = -\frac{1}{y} \left( py^p f(t) + \frac{y'}{y} \right) + \frac{1}{y} \left( py^p f(t) + 2\frac{y'}{y} \right) - \frac{y'}{y^2} = 0 \quad (39)$$

since  $Q_k = 1$ .

### 3.2 Painlevé-Ince V

The divergence of Painlevé-Ince equation V

$$y'' = -2yy' + q(t)y' + q'(t)y \quad (40)$$



yields

$$\lambda_J = -2y + q(t). \quad (41)$$

If we put  $\lambda_J$  into the  $\lambda$ -prolongation then (14) yields two  $\lambda$ -symmetries, i.e.

$$X_1^{(\lambda)} = \partial_y, \quad X_2^{(\lambda)} = \partial_t + (yq(t) - y^2) \partial_y. \quad (42)$$

These two  $\lambda$ -symmetries (42) are equivalent [23]. In fact

$$Q_1(A)(Q_2) - Q_2(A)(Q_1) = yq'(t) + y'(q(t) - 2y) - (-2yy' + q(t)y' + q'(t)y) = 0 \quad (43)$$

since  $Q_1 = 1$  and  $Q_2 = yq(t) - y^2 - y'$ .

Therefore we consider only one  $\lambda$ -symmetry,  $X_1^{(\lambda)}$ . Its first prolongation, i.e.

$$\text{pr}X_1^{(\lambda)} = X_1^{(\lambda)} + (q(t) - 2y) \partial_{y'} \quad (44)$$

yields the first-order invariants

$$y_1 = -yq(t) + y^2 + y', \quad t_1 = t \quad (45)$$

that replaced into equation (40) generate the first-order equation

$$y'_1 = 0 \implies y_1 = a_1 \implies -yq(t) + y^2 + y' = a_1, \quad (46)$$

and therefore the known first integral of (40) is derived [9].

As far as we know  $X_1^{(\lambda)}$  in (42) is a new  $\lambda$ -symmetry of (40).

### 3.3 Painlevé-Ince XIV

In [22] a  $\lambda$ -symmetry of equation Painlevé-Ince XIV equation (3) was determined by assuming that  $\lambda$  was linear with respect to  $y'$ , and thus the following  $\lambda$ -symmetry was found

$$X^{(\lambda_k)} = \partial_y \quad \text{with} \quad \lambda_k = yQ(t) + \frac{S(t)}{y} + \frac{y'}{y}. \quad (47)$$

Instead the divergence of equation (3) yields

$$\lambda_J = \frac{S(t)}{y} + Q(t)y + D_t(\log(y^2)). \quad (48)$$

If we put  $\lambda_J$  into the  $\lambda$ -prolongation and solve the determining equation (14) we get two  $\lambda$ -symmetries, i.e.

$$X_1^{(\lambda)} = \frac{1}{y} \partial_y, \quad X_2^{(\lambda)} = \frac{1}{y^2} \partial_t + \left( -\frac{S(t)}{y^2} + Q(t) \right) \partial_y. \quad (49)$$

One can prove that these two  $\lambda$ -symmetries and that found in [22] are equivalent. The first prolongation of  $X_1^{(\lambda)}$ , i.e.

$$\text{pr}X_1^{(\lambda)} = X_1^{(\lambda)} + \left( \frac{S(t)}{y^2} + Q(t) + \frac{y'}{y^2} \right) \partial_{y'} \quad (50)$$

yields the first-order invariants

$$y_1 = \frac{S(t)}{y} - Q(t)y + \frac{y'}{y}, \quad t_1 = t \quad (51)$$

that replaced into equation (3) generate the first-order equation

$$y'_1 = 0 \implies y_1 = a_1 \implies \frac{S(t)}{y} - Q(t)y + \frac{y'}{y} = a_1, \quad (52)$$

and thus the known first integral (4) of (3) is derived.

### 3.4 Painlevé-Ince XV

Painlevé-Ince XV equation

$$y'' = \frac{y'^2}{y} + \frac{y'}{y} + r(t)y^2 - y \frac{d}{dt} \left( \frac{r'(t)}{r(t)} \right) \quad (53)$$

is known to possess a first integral [9], i.e.:

$$\frac{1}{y^2} \left( \frac{r'(t)}{r(t)} y + y' + 1 \right)^2 - 2 \left( r(t)y + \int r(t) dt \right) = a_1. \quad (54)$$

The divergence of Painlevé-Ince XV equation (53) yields

$$\lambda_J = \frac{1}{y} + D_t (\log(y^2)). \quad (55)$$

If we put  $\lambda_J$  into (12) then (14) yields one  $\lambda$ -symmetry, i.e.

$$X^{(\lambda)} = \frac{1}{y^2} \partial_t - \frac{r'(t)y + r(t)}{r(t)y^2} \partial_y. \quad (56)$$

The first prolongation of  $X^{(\lambda)}$ , i.e.

$$\text{pr}X^{(\lambda)} = X^{(\lambda)} + \frac{-r''(t)r(t)y^2 + r'(t)^2y^2 - r'(t)r(t)y(y' + 1) - r(t)^2(y' + 1)}{r(t)^2y^3} \partial_{y'} \quad (57)$$

yields the first-order invariants

$$\tilde{y} = \frac{r'(t)}{r(t)} + \frac{y' + 1}{y}, \quad \tilde{t} = r(t)y + \int r(t) dt \quad (58)$$

that replaced into equation (53) generate the first-order equation

$$\frac{d\tilde{y}}{d\tilde{t}} = \frac{1}{\tilde{y}} \implies \tilde{y}^2 - 2\tilde{t} = a_1 \implies \left( \frac{r'(t)}{r(t)} + \frac{y' + 1}{y} \right)^2 - 2 \left( r(t)y + \int r(t)dt \right) = a_1, \quad (59)$$

and thus the known first integral (54) of (53) is derived.

As far as we know  $X^{(\lambda)}$  is a novel  $\lambda$ -symmetry of equation (60).

### 3.5 Painlevé-Ince XVI

Painlevé-Ince XVI equation

$$y'' = \frac{y'^2}{y} - q'(t)\frac{y'}{y} + y^3 - q(t)y^2 + q''(t) \quad (60)$$

is known to possess a first integral [9], i.e.:

$$\left( \frac{y' - q'(t)}{y} \right)^2 - (y - q(t))^2 = a_1. \quad (61)$$

The divergence of Painlevé-Ince XVI equation (60) yields

$$\lambda_J = -\frac{q'(t)}{y} + D_t (\log(y^2)). \quad (62)$$

If we put  $\lambda_J$  into (12) then (14) yields one  $\lambda$ -symmetry, i.e.

$$X^{(\lambda)} = \frac{1}{y^2} \partial_t + \frac{q'(t)}{y^2} \partial_y. \quad (63)$$

The first prolongation of  $X^{(\lambda)}$ , (12), i.e.

$$\text{pr}X^{(\lambda)} = X^{(\lambda)} + \frac{q''(t)y - q'(t)^2 + q'(t)y'}{y^3} \partial_{y'} \quad (64)$$

yields the first-order invariants

$$\tilde{y} = \frac{y' - q'(t)}{y}, \quad \tilde{t} = y - q(t) \quad (65)$$

that replaced into equation (60) generate the first-order equation

$$\frac{d\tilde{y}}{d\tilde{t}} = \frac{\tilde{t}}{\tilde{y}} \implies \tilde{y}^2 - \tilde{t}^2 = a_1 \implies \left( \frac{y' - q'(t)}{y} \right)^2 - (y - q(t))^2 = a_1, \quad (66)$$

and thus the known first integral (61) of (60) is derived. As far as we know  $X^{(\lambda)}$  is a novel  $\lambda$ -symmetry of equation (60).

### 3.6 Example 4 in [3]

In [3] a  $\lambda$ -symmetry of equation

$$y'' = (ty' - ty^2 + y^2) \exp(-1/y) + 2\frac{y'^2}{y} + y' \quad (67)$$

was determined with  $\lambda_k = t \exp(-1/y) - 1/t$ . This equation has been completely solved by considering its solvable structures [3].

The divergence of equation (67)– that has no point symmetries – yields

$$\lambda_J = t \exp(-1/y) + D_t (\log(y^4) + t). \quad (68)$$

If we put  $\lambda_J$  into the  $\lambda$ -prolongation and solve the determining equations (14) we get two  $\lambda$ -symmetries, i.e.

$$X_1^{(\lambda)} = \frac{1}{y^2 \exp(t)} \partial_y, \quad X_2^{(\lambda)} = \frac{1}{y^2 \exp(2t)} \left( \frac{1}{y^2} \partial_t + \frac{t}{\exp(1/y)} \partial_y \right). \quad (69)$$

One can prove that these two  $\lambda$ -symmetries and that found in [3] are equivalent. The first prolongation of  $X_1^{(\lambda)}$ , i.e.

$$\text{pr}X_1^{(\lambda)} = X_1^{(\lambda)} + \left( \frac{2y'}{y^3 \exp(t)} + \frac{t}{y^2 \exp(t + 1/y)} \right) \partial_{y'} \quad (70)$$

yields the first-order invariants

$$y_1 = \frac{y'}{y^2} - t \exp(-1/y), \quad t_1 = t \quad (71)$$

that replaced into equation (67) generate the first-order equation

$$y'_1 = y_1 \implies y_1 = a_1 \exp(t) \implies \left( \frac{y'}{y^2} - t \exp(-1/y) \right) \exp(-t) = a_1, \quad (72)$$

and therefore a first integral of (67) is derived.

### 3.7 Example 5 in [3]

In [3] a  $\lambda$ -symmetry of equation

$$y'' = 2\frac{y'^2}{y} + \left( t \exp(t/y) - \frac{4}{t} \right) y' - \left( 3\frac{y^2}{t} + y \right) \exp(t/y) + ty^2 + 2\frac{y}{t^2} \quad (73)$$

was determined with  $\lambda_k = t \exp(t/y)$ . The divergence of equation (73)– that has no point symmetries – yields

$$\lambda_J = t \exp(t/y) + D_t \left( \log \left( \frac{y^4}{t^4} \right) \right). \quad (74)$$

If we put  $\lambda_J$  into (12) then (14) yields two  $\lambda$ -symmetries, i.e.

$$X_1^{(\lambda)} = \frac{t^3}{y^2} \partial_y, \quad X_2^{(\lambda)} = \frac{t^6}{y^4} \partial_t + \frac{-5t^6 y \exp(t/y) + t^8 y + 5t^5}{5y^3} \partial_y. \quad (75)$$

One can prove that these two  $\lambda$ -symmetries and that found in [3] are equivalent. The first prolongation of  $X_1^{(\lambda)}$ , i.e.

$$\text{pr} X_1^{(\lambda)} = X_1^{(\lambda)} + \frac{t^2}{y^3} (t^2 y \exp(t/y) - y + 2ty') \partial_{y'} \quad (76)$$

yields the first-order invariants

$$y_1 = \frac{ty^2 \exp(t/y) - y + ty'}{ty^2}, \quad t_1 = t \quad (77)$$

that replaced into equation (73) generate the first-order equation

$$y_1' = \frac{-3y_1 + t^2}{t} \implies y_1 = \frac{5a_1 + t^5}{5t^3} \implies t^2 \frac{ty^2 \exp(t/y) - y + ty'}{y^2} - \frac{t^5}{5} = a_1, \quad (78)$$

and therefore a first integral of (73) is derived.

## 4 Conclusions

In this paper we have shown that the Jacobi last multiplier provide and algorithmic simple way to construct  $\lambda$ -symmetries. Once a  $\lambda$ -symmetry is obtained then it is a simple task to derive a first integral as we have shown in the many examples presented in Section 3.

We remark that Strategy 3 as described in [29] is not equivalent to the reduction by using  $\lambda$ -symmetries. In fact different output are obtained. For example, equation (1) was completely solved by quadrature using Strategy 3, while the reduction by using  $\lambda$ -symmetries yields just a first integral. Yet those two methods may complement each other as in the example of equation (2).

Also one should be aware of the fact that a  $\lambda$ -symmetry could be equivalent to a Lie point symmetry. Lie point symmetries may be obviously considered  $\lambda$ -symmetries with  $\lambda = 0$ . For example, let us consider equation (38) in [22], i.e.

$$2yy'' - 6y'^2 + y^5 + y^2 = 0, \quad (79)$$

which admits a trivial Lie point symmetry, i.e.  $\Gamma = \partial_t$ . The divergence of this equation yields

$$\lambda_J = 6 \frac{y'}{y} = D_t(\log(y^6)). \quad (80)$$

If we put  $\lambda_J$  into (12) then (14) yields one  $\lambda$ -symmetry, i.e.

$$X^{(\lambda)} = \frac{1}{y^6} \partial_t. \quad (81)$$

It is easy to prove that this  $\lambda$ -symmetry is equivalent to the Lie point symmetry  $\Gamma$ .

We now search for  $\lambda$ -symmetries of equation (27) which, if  $A = a$ , admits two Lie point symmetries, i.e.  $\Gamma_1 = \partial_t$ ,  $\Gamma_2 = \exp(-at)(\partial_t + a\partial_{r_2})$ . The divergence of equation (27) yields

$$\lambda_J = b \exp(r_2) + a. \quad (82)$$

If we put  $\lambda_J$  into (12) then (14) yields one  $\lambda$ -symmetry, i.e.

$$X^{(\lambda)} = \exp(-at) \partial_{r_2}. \quad (83)$$

It is easy to prove that this  $\lambda$ -symmetry is equivalent to the Lie point symmetry  $\Gamma_2$ .

A more detailed analysis of the Jacobi last multiplier approach to  $\lambda$ -symmetries is needed. In particular the analysis of higher order ODEs can provide new ideas and confirm the importance of this method for the integration of ODEs. Moreover its extension to PDEs can provide new insights on the meaning of  $\mu$ -symmetries [4], [6].

Work is in progress in these directions.

## Acknowledgements

This work was initiated while DL was enjoying the hospitality of MCN. DL thanks the Dipartimento di Matematica e Informatica, Università di Perugia, for the provision of facilities. DL has been partly supported by the Italian Ministry of Education and Research, 2010 PRIN “Continuous and discrete nonlinear integrable evolutions: from water waves to symplectic maps”.

## References

- [1] L. Bianchi, *Lezioni sulla teoria dei gruppi continui finiti di trasformazioni*, Enrico Spoerri, Pisa (1918).
- [2] D. Catalano Ferraioli, Nonlocal aspects of  $\lambda$ -symmetries and ODEs reduction, *J. Phys. A: Math. Theor.* **40**, 5479-5489 (2007).
- [3] D. Catalano Ferraioli, P. Morando, Local and nonlocal solvable structures in the reduction of ODEs, *J. Phys. A: Math. Theor.* **42**, 035210 (2009).
- [4] G. Cicogna, G. Gaeta, P. Morando, On the relation between standard and  $\mu$ -symmetries for PDEs, *J. Phys. A: Math. Theor.* **37**, 9467-9486 (2004).

- [5] M. Edwards, M.C. Nucci, Application of Lie group analysis to a core group model for sexually transmitted diseases, *J. Nonlinear Math. Phys.* **13**, 211-230 (2006).
- [6] G. Gaeta, P. Morando, On the geometry of  $\lambda$ -symmetries and PDE reduction, *J. Phys. A: Math. Theor.* **37**, 6955-6975 (2004).
- [7] A. González-López, Symmetries of linear systems of second-order ordinary differential equations, *J. Math. Phys.* **29**, 1097-1105 (1988).
- [8] A. Gradassi, M.C. Nucci, Hidden linearity in systems for competition with evolution in ecology and finance, *J. Math. Anal. Appl.* **333**, 274-294 (2007).
- [9] E.L. Ince, *Ordinary Differential Equations*, Dover, New York (1956).
- [10] P. E. Hydon, Symmetry analysis of initial-value problems, *J. Math. Anal. Appl.* **309**, 103-116 (2005).
- [11] C.G.J. Jacobi, Sul principio dell'ultimo moltiplicatore, e suo uso come nuovo principio generale di meccanica, *Giornale Arcadico di Scienze, Lettere ed Arti* **Tomo 99**, 129-146 (1844).
- [12] C.G.J. Jacobi, Theoria novi multiplicatoris systemati æquationum differentialum vulgarium applicandi: Pars I *J. Reine Angew. Math.* **27**, 199-268 (1844).
- [13] C.G.J. Jacobi, Theoria novi multiplicatoris systemati æquationum differentialum vulgarium applicandi: Pars II *J. Reine Angew. Math.* **29**, 213-279 and 333-376 (1845).
- [14] C.G.J. Jacobi, *Vorlesungen über Dynamik. Nebst fünf hinterlassenen Abhandlungen desselben herausgegeben von A Clebsch*, Druck und Verlag von Georg Reimer, Berlin (1886).
- [15] E. Kamke, *Differentialgleichungen Lösungsmethoden und Lösungen*, Chelsea, New York (1974).
- [16] P.G.L. Leach, M.C. Nucci, Reduction of the classical MICZ-Kepler problem to a two-dimensional linear isotropic harmonic oscillator, *J. Math. Phys.* **45**, 3590-3604 (2004).
- [17] S. Lie, Veralgemeinerung und neue Verwerthung der Jacobischen Multiplicator-Theorie *Fordhandlingar i Videnokabs - Selshabet i Christiania*, 255-274 (1874).
- [18] S. Lie, *Vorlesungen über Differentialgleichungen mit bekannten infinitesimalen Transformationen*, Teubner, Leipzig (1912).
- [19] M. Marcelli, M.C. Nucci, Lie point symmetries and first integrals: the Kowalevsky top, *J. Math. Phys.* **44**, 2111-2132 (2003).

- [20] C. Muriel, J.L. Romero, New methods of reduction for ordinary differential equations, *IMA J. Appl. Math.* **66**, 111-125 (2001).
- [21] C. Muriel, J.L. Romero,  $C^\infty$ -symmetries and reduction of equations without Lie point symmetries, *J. Lie Theory* **13**, 167-188 (2003).
- [22] C. Muriel, J.L. Romero, First integrals, integrating factors and  $\lambda$ -symmetries of second-order differential equations, *J. Phys. A: Math. Theor.* **42** 365207 (2009).
- [23] C. Muriel, J.L. Romero,  $\lambda$ -symmetries on the derivation of first integrals of ordinary differential equations, *WASCOM 2009: 15th Conf. on Waves and Stability in Continuous Media*, A. M. Greco, S. Rionero, T. Ruggeri eds., World Scientific, New York (2010) pp. 303-308.
- [24] E. Noether, Invariante Variationsprobleme, *Nachr d König Gesellsch d Wiss zu Göttingen, Math-phys Klasse*, 235-257 (1918).
- [25] M.C. Nucci, Interactive REDUCE programs for calculating Lie point, non-classical, Lie-Bäcklund, and approximate symmetries of differential equations: manual and floppy disk, in *CRC Handbook of Lie Group Analysis of Differential Equations. Vol. 3: New Trends in Theoretical Developments and Computational Methods*, N.H. Ibragimov ed., CRC Press, Boca Raton (1996) pp. 415–481.
- [26] M.C. Nucci, The complete Kepler group can be derived by Lie group analysis, *J. Math. Phys.* **37**, 772-1775 (1996).
- [27] M.C. Nucci, Lorenz integrable system moves à la Poincaré, *J. Math. Phys.* **44**, 4107-4118 (2003).
- [28] M.C. Nucci, Calogero’s “goldfish” is indeed a school of free particles, *J. Phys. A: Math. Gen.* **37**, 11391-11400 (2004).
- [29] M.C. Nucci, Jacobi last multiplier and Lie symmetries: a novel application of an old relationship, *J. Nonlinear Math. Phys.* **12**, 284-304 (2005).
- [30] M.C. Nucci, Lie symmetries of a Painlevé-type equation without Lie symmetries, *J. Nonlinear Math. Phys.* **15**, 205-211(2008).
- [31] M.C. Nucci, P.G.L. Leach, The harmony in the Kepler and related problems, *J. Math. Phys.* **42**, 746-764 (2001).
- [32] M.C. Nucci, P.G.L. Leach, An integrable S-I-S model, *J. Math. Anal. Appl.* **290** (2004), 506–518.
- [33] M.C. Nucci and K.M. Tamizhmani, “Lagrangians for biological models” arXiv:1108.2301 (2011) submitted



- [34] P. Painlevé, Mémoire sur les équations différentielles dont l'intégrale générale est uniforme, *Bull. Soc. Math. Fr.* **28**, 201-261 (1900).
- [35] E. Pucci, G. Saccomandi, On the reduction methods for ordinary differential equations, *J. Phys. A: Math. Gen.* **35**, 6145-6155 (2002).
- [36] V. Torrisi, M.C. Nucci, Application of Lie group analysis to a mathematical model which describes HIV transmission, pp. 31-40, *The Geometrical Study of Differential Equations*, J. A. Leslie, and T. P. Hobart, eds., AMS, Providence (2001).
- [37] E.T. Whittaker, *A Treatise on the Analytical Dynamics of Particles and Rigid Bodies*, Cambridge University Press, Cambridge, 1988, first published in 1904.